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# THE BEHAVIOR UNDER PROJECTION OF DILATING SETS IN A COVERING SPACE

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ABSTRACT. Let $M$ be a compact Riemannian manifold with covering space $S$, and suppose $d\mu_r$ $(0 < r < \infty)$ is a family of Borel probability measures on $S$, all of which arise from some fixed measure by $r$-homotheties of $S$ about some point, followed by renormalization of the resulting measure. In this paper we study the ergodic properties, as a function of $r$, of the corresponding family of projected measures on $M$ in the Euclidean and hyperbolic cases. A typical example arises by considering the behavior of a dilating family of spheres under projection.

Suppose $T^n$ is the $n$-dimensional integral torus, regarded as the quotient of $R^n$ by the lattice $\Gamma$ of integral translations, and suppose $d\mu$ is a Borel probability measure on $R^n$, i.e., $d\mu$ is a real, nonnegative Borel measure on $R^n$ having total mass 1. By projection, $d\mu$ gives rise to a Borel probability measure $dm$ on $T^n$, if we define the measure of a set in $T^n$ to be the measure with respect to $d\mu$ of its total preimage in $R^n$ with respect to the projection map from $R^n$ to $T^n$.

For various types of $\Gamma$-invariant functions $f$ on $R^n$, or, what is the same thing, functions on $T^n$, it is natural to study the difference

$$\int_{T^n} f(x)\,dx - \int_{T^n} f(x)\,dm(x), \tag{1}$$

where $dx$ is normalized Lebesgue measure. This quantity arises, for example, in the study of equidistribution of sequences, and in the study of ergodic properties of irrational lines in $T^n$.

In order to study (1), suppose, for the moment, that $f$ is $C^\infty$ with Fourier series $\sum c_N\, e^{2\pi i(N,x)}$, and suppose $\Phi(y) = \int e^{2\pi i(x,y)}\, d\mu(x)$ is the Fourier transform of $d\mu$. Then it is easily checked that the $N$th Fourier coefficient of $dm(x)$ is $\Phi(-N)$, and since $f$ is smooth the Parseval equality can be applied to conclude that

$$\int_{T^n} f(x)\,dm(x) = \sum c_N\,\Phi(N) = c_0 + {\sum}' c_N \Phi(N),$$

where the prime indicates that the origin is omitted from the summation.

But $c_0 = \int_{T^n} f(x)\,dx$, so

$$\int_{T^n} f(x)\,dx - \int_{T^n} f(x)\,dm(x) = {\sum}' c_N \Phi(N). \tag{2}$$

I.e., from our point of view, the study of (1) is equivalent to the study of the asymptotic behavior of $\Phi$. It is the purpose of this paper to illustrate this principle

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in a few, by no means exhaustive, cases. In particular, in what follows we will be concerned with the situation in which there is a family $d\mu_r$ $(0 < r < \infty)$ of probability measures on $R^n$, all of which arise from some fixed measure by $r$-homotheties of the underlying space, followed by renormalization of the resulting measure.

It is a pleasure to thank Dennis Sullivan for a conversation which gave rise to the present paper, in which he asked what one could say about the ergodic properties of the projection of a progressively dilating curvilinear body on $T^n$, and we begin with a theorem which deals with this question.

THEOREM 1. *Suppose $C$ is the smooth boundary of a compact convex $n$-dimensional body in $R^n$, and suppose its Gaussian curvature, i.e., the Radon-Nikodym derivative of the Gauss map, is everywhere positive. For a continuous $\Gamma$-invariant function $f$ on $R^n$ having Fourier coefficients $\{c_N\}$, define, as suggested by* (1),

$$D_r(f,C) = \int_{T^n} f(x)\,dx - \frac{r^{-(n-1)}}{\text{area}(C)} \int_{rC} f(x)\,ds_x,$$

*where $ds_x = (n-1)$-dimensional Lebesgue measure. Then if $\sum' |c_N|\,|N|^{-(n-1)/2}$ converges, we have*

$$D_r(f,C) = O(r^{-(n-1)/2}).$$

REMARKS. 1. The convergence criterion is certainly satisfied if, for example, $f$ is $C^m$, for $m > (n-1)/2$.

2. This theorem shows that subject to the convergence criterion, the projected mass is equidistributed in the limit.

3. It is possible to drop the assumption of strictly positive curvature in many cases, although at the cost of increased intricacy in the statements and proofs of the corresponding theorems (cf. [**5**, **6**]). One can also treat the case of the dilation of a $(k-1)$-dimensional hypersurface lying in a $k$-plane of $R^n$, or for that matter, the case of a dilating solid body. Rather than enter into a general discussion of these questions at this time, we will, in Theorem 2, take up the important and relatively easily handled special case of the dilation of a rectilinear $k$ simplex $(1 \le k \le n)$.

PROOF OF THEOREM 1. Assume for the moment that $f$ is $C^\infty$. Denote by $\Phi(y)$ the Fourier transform of normalized Lebesgue measure on $C$, and by $\Phi_r(y)$ the Fourier transform of normalized Lebesgue measure on $rC$. Then

$$\begin{aligned}\Phi_r(y) &= \frac{r^{-(n-1)}}{\text{area}(C)} \int_{rC} e^{2\pi i(x,y)}\,dx_x \\ &= \frac{1}{\text{area}(C)} \int_C e^{2\pi i(rx,y)}\,ds_x \\ &= \Phi(ry),\end{aligned}$$

since the $r$ can be shifted to $y$. Therefore by (2),

$$D_r(f,C) = \sum{}' c_N \Phi(rN).$$

Now under the hypotheses on $\partial C$, it is well known [**4**] that $\Phi(y) = O(|y|^{-(n-1)/2})$, and the result follows immediately if $f$ is $C^\infty$. If $f$ is merely continuous, and $\sum' |c_N|\;|N|^{-(n-1)/2}$ is convergent, we convolve $f$ with an approximate $\delta$-function

$\delta_\varepsilon(x)$ on $T^n$, which is $C^\infty$, nonnegative, supported in the ball of radius $\varepsilon$, and has integral 1. Set $f_\varepsilon = f * \delta_\varepsilon$. Then clearly for any fixed $r$, $D_r(f_\varepsilon, C) \to D_r(f, C)$ as $\varepsilon \to 0$. Moreover if we denote the Fourier coefficients of $f_\varepsilon$ by $c_N(\varepsilon)$, then $c_N(\varepsilon) = c_N c_N^{\#}(\varepsilon)$, where $c_N^{\#}(\varepsilon)$ is the $N$th Fourier coefficient of $\delta_\varepsilon(x)$. But $|c_N^{\#}(\varepsilon)| \leq 1$ and for each $N$, $c_N^{\#}(\varepsilon) \to 1$ as $\varepsilon \to 0$. It follows from the dominated convergence theorem that $D_r(f, C) = \sum' c_N \Phi(rN)$, which proves the theorem.

In order to prove Theorem 2, which treats the case of a dilating rectilinear $k$-simplex ($1 \leq k \leq n$), we first need to develop some simple facts about the Fourier transform of such a simplex (cf. [**5**, **11**]).

Suppose $C$ is a rectilinear $k$-simplex in $R^n$, which we may assume contains the origin, since this can always be brought about by a translation, which has the effect of multiplying the Fourier transform by a character of absolute value 1. Then it is easy to see that the Fourier transform $\Phi$ of $C$ can be expressed in the form

$$\Phi(y) = \int_C e^{2\pi i (x, P(y))}\, ds_k(x), \tag{3}$$

where $ds_k(x)$ is $k$-dimensional Lebesgue measure on the $k$-plane $S_k$ containing $C$, and $P(y)$ is the projection of $y$ on $S_k$. By the divergence theorem, (3) is equal to

$$\frac{1}{2\pi i |P(y)|} \int_{\partial C} e^{2\pi i (x, P(y))} (\theta(y), n(x))\, ds_{k-1}(x),$$

where $\theta(y) = (y)/|P(y)|$, $ds_{k-1}(x)$ is $(k-1)$-dimensional Lebesgue measure on $\partial C$, and $n(x)$ is the external normal to $\partial C$ in $S_k$. Note that $(\theta(y), n(x))$ is a constant of absolute value $\leq 1$ on each face of $\partial C$.

By repeating the above argument, with $C$ replaced by a $(k-1)$-face of $C$, and continuing in this fashion until we end with 0-dimensional faces, we ultimately see that $\Phi(y)$ can be expressed as a sum of terms, each one of which is of the general form

$$c(y)(2\pi i)^{-k} \prod_{j=1}^{k} |P_j \circ P_{j+1} \circ \cdots \circ P_k(y)|^{-1} \tag{4}$$

where $|c(y)| \leq 1$, and $P_1, \ldots, P_k = P$ is a family of projections onto a nested collection $S_1 \subset S_2 \subset \cdots \subset S_k$ of subspaces of $R^n$ having dimensions $1, \ldots, k$, respectively (cf. [**5**, **10**]).

Since the $S_j$'s are nested, $P_m \circ P_n = P_{\min(m,n)}$, so (4) becomes

$$c(y)(2\pi i)^{-k} \prod_{j=1}^{k} |P_j(y)|^{-1}. \tag{5}$$

THEOREM 2. *Suppose $f$ is a continuous $\Gamma$-invariant function having Fourier coefficients $\{c_N\}$. Keeping notation as in Theorem* 1, *suppose $C$ is a $k$-dimensional rectilinear simplex in $R^n$, and suppose, for any flag $S_1 \subset S_2 \subset \cdots \subset S_k$ of subspaces of $R^n$ generated by $C$ in the previously discussed manner, with associated projections $P_1, \ldots, P_k$, that the series*

$$\sum_N{}' |c_N| \left( \prod_{j=1}^{k} |P_j(N)|^{-1} \right)$$

*is convergent. Then $D_r(f,C)$ can be expressed as a finite sum of terms, each one of which is of the form*

$$c(rN)r^{-k}\sum_N{}' c_N \prod_{j=1}^k |P_j(N)|^{-1},$$

*and arises from a flag $S_1 \subset \cdots \subset S_k$ in the previously discussed manner. In particular,*

$$D_r(f,C) = O(r^{-k}).$$

PROOF. As in the proof of Theorem 1, note that $\Phi_r(y) = \Phi(ry)$, so, for smooth $f$, we have, by (2), $D_r(f,C) = \sum' c_N \Phi(rN)$. By (5) this proves the theorem for smooth $f$. The proof for general continuous $f$ is them completed by the method used in the proof of Theorem 1.

EXAMPLE. Suppose $C$ is a 1-dimensional segment in $R^n$, and suppose $(x_1, \ldots, x_n)$ is an orthonormal coordinate system such that $\Gamma$ consists of points having integral coordinates. Note that in this case there is only one flag, which contains a single 1-dimensional subspace, and only one associated projection $P_1$. Let $S$ be any 2-dimensional subspace of $R^n$ defined by the vanishing of all but two of the $x_j$'s, and let $C'$ be the projection of $C$ onto $S$. Then it follows easily from Roth's theorem (cf. [**10**]) that if $C'$, regarded as a line within $S$, has an irrational algebraic slope, then for any $\varepsilon > 0$, $|P_1(N)| > |N|^{-(1+\varepsilon)}$ for all but a finite number of lattice-points in $R^n$. If therefore $f$ is a continuous $\Gamma$-invariant function with $c_N = O(|N|^{-(n+1+\delta)})$ for some $\delta > 0$, Theorem 2 implies that $D_r(f,C) = O(r^{-1})$.

We conclude with some remarks about the hyperbolic case. In this case the role of $R^n$ is played by $n$-dimensional hyperbolic space $H^n$, and $\Gamma$ is a discontinuous group acting on $H^n$. Additionally, it is natural to assume that the measure being dilated has spherical symmetry around the point $x$ about which the dilation occurs. We will briefly illustrate things for the typical case in which $d\mu_T$ is normalized Riemannian measure on a sphere of radius $T$ about a point $x \in H^3$, and $\Gamma$ is a torsion-free cocompact lattice. The role of $C$ is then played by the unit sphere in $H^n$. The treatment of the hyperbolic case in all dimensions is essentially the same, but the transforms which arise in $H^3$ are particularly simple. We remark that Peter Sarnak has also studied the case of a dilating sphere in $H^3$ using the wave equation.

Assume for simplicity that $f$ is $C^\infty$ on $M = \Gamma\backslash H^3$ and has the Fourier expansion $\sum c_n\varphi_n$, where $\varphi_0, \varphi_1, \ldots$ is a complete orthonormal set of eigenfunctions for the Laplacian on $M$, corresponding to eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$. (Note that $\varphi_0(x) \equiv A^{-1/2}$, where $A$ is the volume of $M$. Note also that $c_0 = A^{-1/2}\int_M f$.)

If, now, $d\mu_T(x,y)$ is normalized hyperbolic area measure on the sphere of radius $T$ about a fixed point $x \in H^3$, then the projected measure $dm_T(x,y)$ on $M$ will be given by

$$dm_T(x,y) = \sum_{\gamma\in\Gamma} d\mu_T(x,\gamma y). \tag{6}$$

It is easy to analyze the right-hand side of (6) using techniques connected with the Selberg trace formula [**3**, **8**], and we find that we can compute the effect of $dm_T(x,y)$ on a $C^\infty$ function on $M$ by the formula

$$\int_C f(y)\, dm_T(x,y) = \frac{1}{2}\sum c_n h_T(r_n)\varphi_n(x),$$

where there are two $r_n$'s for each eigenvalue $\lambda_n$, defined by $\lambda_n = 1 + r_n^2$ (if $r_n = 0$ is present it is counted with multiplicity 2), and the function $h_T$ is given by $h_T(u) = (\sin uT)(u \sinh T)^{-1}$ [**1**, **2**, **9**].

Now

$$\begin{aligned} c_n &= (f, \varphi_n) \\ &= \lambda_n^{-i}(f, \Delta^i \varphi_n) \quad \text{for any } i \geq 1 \\ &= \lambda_n^{-i}(\Delta^i f, \varphi_n) \leq \lambda_n^{-i} \|\Delta^i f\|_2 \end{aligned}$$

by the Cauchy-Schwarz inequality.

On the other hand, it is well known that for $j$ large enough a fundamental solution for $\Delta^j$ on $M$ is given by an integral operator with continuous kernel $K(x, y)$. Since $\Delta^j \varphi_n = \lambda_n^j \varphi_n$, this implies

$$\varphi_n(x) = \lambda_n^j \int_M K(x, y) \varphi_n(y)\, dy,$$

so again by the Cauchy-Schwarz inequality, $|\varphi_n(x)| \leq c\lambda_n^j$ for some fixed $j$. By taking $i$ sufficiently large, we see that given $p > 0$, the series $\sum c_n |\varphi_n(x)|$ is eventually dominated by the series $\sum \lambda_n^{-p}$, and since the latter is convergent for large $p$, we conclude that for any $x$, $\sum c_n |\varphi_n(x)|$ is convergent. Since the two $r_n$'s corresponding to $\lambda_0 = 0$ are $i$ and $-i$, and since $c_0 h_T(\pm i)\varphi_0(x) = A^{-1} \int_M f$, which is the integral of $f$ over $M$ with respect to normalized measure, we conclude that the quantity $D_T(f, C)$, defined as before, is given by $\frac{1}{2} \sum' c_n h_T(r_n) \varphi_n(x)$, where the prime means that the values $r_n = \pm i$ are omitted from the sum. In particular, if $M$ has small eigenvalues, i.e., if $\lambda_1 \in (0, 1)$, then

$$D_T(f, M) = O((\sin r_1 T)(\sinh T)^{-1}) = O(e^{-\alpha T}),$$

where $\alpha = 1 - |r_1|$, with $\lambda_1 = 1 + r_1^2$.

If $M$ has no small eigenvalues, then $D_T(f, M) = O(e^{-T})$.

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